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THE HIGHER THEORY OF STATISTICS.

Die Grundzüge der Theorie der Statistik. Von Harald Westergaard, Professor an der Universität zu Kopenhagen. (Jena: Verlag von Gustav Fischer.)

THIS is an important contribution to the Calculus of Probabilities and the higher theory of Statistics. The foundation of experience on which the whole edifice of probabilities is based has been strengthened and extended by the new material which Prof. Westergaard has deposited. Here, for instance, is one of his experiments:—From a bag containing black and white balls in equal numbers, he drew (or caused to be drawn) a ball 10,000 times, the ball being replaced in the bag and the bag shaken up after each extraction. He records not only the total numbers of each colour, but also the number of white balls in each of 100 batches, each numbering 100 balls, also in 50 batches each of 200 balls, and so on. The diminution of the relative deviation from the average as the size of the batch is increased comes out clearly. On an equally large scale Prof. Westergaard has observed the proportion of prizes to blanks in batches of tickets drawn at a lottery; and the frequency with which different numbers, drawn under conditions such that one number was as likely as another, were observed to occur actually. He has similarly tabulated the frequency with which the different digits 1, 2, 3, &c., terminate certain officially recorded amounts, the "kontis" of a savings bank, of which documents he has examined 10,000. These experiences afford new confirmation to the first principles of the calculus: namely, the fundamental fact of statistical regularity which the definition of probability involves, and the postulate that certain events are independent of each other in such wise that, if the probability of each be , the probability of the double event is a quarter.

Ascending from these simple experiences, Prof. Westergaard reaches by a new and easy route the formula for the measure, or *modulus*, of the extent to which the observed number—*e.g.* of white balls in a batch of 100 or 1000—is likely to differ from the most probable number; in the instance just given $100p$ or $1000p$, if p is the probability of drawing a white ball. The sought expression, it is presumed, must be a *symmetric* function of the probability of the event (drawing a white ball), which we have called p, and the complementary probability (drawing a non-white ball), viz., $1-p$. This hint enables us to decipher from the records of experience that the modulus is proportioned to $\sqrt{p(1-p)}$. The influence of the *size* of the batch upon the extent of the deviation is similarly elicited from observation. Thus with a minimum of mathematical equipment, by easy steps and through an unpretentious *a posteriori* gate, we are led into the very stronghold of Probabilities—if not to the law of error itself, at any rate to one of its most important properties.

Prof. Westergaard has not only popularized the law of error, he has also proved it. He has added considerably to its evidence, by observing in an immense number of instances the exact correspondence between fact and theory. We must content ourselves with citing one set of instances. Ten thousand balls having been drawn at random, as above mentioned, and the composition of each batch of a 100 being examined, it was found that for twenty-five out of a hundred such groups the number of white balls lay between 49 and 51 (inclusive)—limits distant ± 1 from 50, the most probable number. For forty out of the hundred groups the number of white balls lay between the limits 50 ± 2. And so on. The observations are exhibited and compared with theory in the annexed table:—

Number of batches in which the number of white balls is between certain limits.		Limits.
Observed.	Calculated.	
25	24	± 1
40	38	± 2
50	52	± 3
70	73	± 5
85	87	± 7
95	96	±10

The multiplication of correspondences like this, the concatenation of evidence in favour of the law of error which the author has put together in his fifth chapter, is very cogent.

Another sort of verification to which the law of error is submitted is to compare it with the explicit binomial to which the exponential law is an approximation. This approximation is closer than may be supposed. For example, if a hundred balls be taken at random, each ball being as likely to be black as white, the probability of obtaining exactly 50 balls, as evaluated by the binomial theorem, is ·080, as approximately determined by the exponential law of error is also ·080. The probability of obtaining either 49, 50, or 51 is, according to the exact calculation, ·236, according to the approximative formula, also ·236. And so on.

Among other contributions to the calculus which Prof. Westergaard has either adduced from authors rarely read, or himself deduced, may be noticed his elegant treatment of the case where the probabilities of two alternative events (say, drawing white or black balls) are not equal (p. 70). Suppose that the probability, say p, of one event is very small, then the formula for the deviation of the number of white balls actually drawn from the number most likely to be drawn, viz. np, admits of simplification. The "mean error" $(= modulus \div \sqrt{2})$ is in general ... $(\sqrt{np(1-p)})$; in the particular case it becomes approximately $\sqrt{np}$. A further simplification may be explained by an example. Suppose that we know the number of deaths, say 900, per unit of time in a certain population. Then, without knowing the number of the population, or without taking the trouble of referring to it and calculating the death-rate, we may determine approximately the fluctuation to which the number of deaths is liable. For the measure of that fluctuation the "mean error," is approximately $\sqrt{np}$; n being the number of population, a large number, and p the death-rate, a small fraction. Now np is 900, and accordingly 30 is the mean error, that is, assuming that the urn in which the lots of Fate are shaken—"omnium Versatur urna serius ocius Sors exitura"—is constituted as simply as the urn in which we have supposed black and white balls to be shaken up. This is a question in Applied Probabilities to which we are just coming.

Prof. Westergaard's applications of the calculus to statistics are even more striking than his developments of the pure theory of probabilities. The law of error may be applied to concrete phenomena in two cases: where the fluctuation of averages follows the analogy of the simpler games of chance—as we just now assumed with regard to deaths—and where this condition is not fulfilled. Prof. Westergaard's contributions belong chiefly to the first class. He has considerably added to the instances discovered by Prof. Lexis, in which a set of ratios—such as the proportion between the mortality of male and female infants in different years—are grouped according to the same law of dispersion as the percentages of white balls in a set of batches drawn at random from an urn containing black and white balls mixed up in a certain proportion. The uses of this discovery are twofold—negative and positive. In the first place, we may be deterred from a search after causes which is hopeless. In the typical instance of the urn and balls it would be vain to trace the reason why any particular ball, or set of balls, extracted should be white (or black). We cannot hope to analyse the "fleeting mass of causes"—as Mill calls Chance—upon which the event depends. We may have been able to break up our batches of balls into two classes, say rough and smooth, such that the rough balls are extracted from an urn containing mostly white, while the smooth balls are more frequently black. But when this process of "depouillement" has been carried as far as possible, when we have reached the ideal type of a single urn and constant proportions, then the investigation of causes halts. Then it is only crazy gamblers who hope to discover a principle underlying the "runs" of black and white balls. We have reached the bounds of the territory of science; beyond there is only the sea of chance. Prof. Westergaard has not only indicated this limit, but also pushed many of his investigations up to it.

These considerations do not preclude us from applying the theory of error to detect delicate distinctions such as the difference between a loaded and a perfect die, which make themselves felt in the averages of great numbers of observations. In fact, it is by the mathematical method that we can best determine whether a difference between two averages is significant of a real constant difference, or only apparent and accidental. Prof. Lexis, and Dr. Duesing after him had applied this method to the investigation of the conditions under which the excess of male over female births is greater or smaller than usual. Prof. Westergaard shows that the method is applicable to many other subjects, among which the mortality at different age-periods promises to be most useful. We could wish, indeed, for more copious evidence in favour of the premise that the mortality of a population at a certain age-period (say of clergymen or innkeepers between the ages 35-44. See *Grundzüge*, p. 82, with context, and *cf.* p. 52) fluctuates according to the analogy of games of chance.

Here the question arises: must the phenomenon under consideration be known to vary after the manner of balls extracted from an urn, in order that the mathematical method may be applicable? Certainly the apparatus of the law of error—probable and improbable deviation—may be employed to ascertain whether the difference between the average heights of two groups of two men is significant or accidental; though in this case the modulus (or mean error) does not follow the analogy of the simpler games of chance. Might we not similarly compare the *general* mortality of two sections of population, although the dispersion of such death-rates about their average is much greater than it should be on the hypothesis of pure sortition. The advantage, indeed, which we have above distinguished as negative, would no longer exist in this case. But might not the positive advantage still be enjoyed in some degree? Prof. Westergaard, so far as we have observed, is silent on this topic.

We have left ourselves too little space for noticing Prof. Westergaard's other contributions to applied Probabilities. His treatment of Insurance, together with the adjacent theory of Life-Tables, involving the arts of Interpolation, may dispute with Cournot's classical chapters the honour of forming the best introduction to the subject for the general reader—the reader prepared by a general mathematical, as distinguished from a special actuarial training. Nor must we pass over the chapter in which the author surveys "economic" (exclusive of Vital) statistics. He has here occasion to employ largely the important principle of inference from samples. For instance, in order to discover the amount of wood in a country, we should first select one or more sample surfaces (*Probe-flächen*), and a sufficient number of sample trees thereat, and then measure the quantity of wood on those trees. "From the figures so found conclusions can be drawn as to the whole sample-surface, and from those to the total quantity of wood in the country." So in order to determine the quantity of milk, we must proceed by way of *Probe-kühe*. The method of samples is no doubt a potent instrument when wielded by a trained hand like Prof. Westergaard's. We may perhaps extend to economics generally what he suggests with reference to its statistical side: that a given effort and expense may be better laid out in obtaining a detailed knowledge of a few parts with a general view of the whole, rather than a more uniformly distributed information.

The last part of the work is devoted to a history of statistics; not a chronicle, but such a history as a great tactician would write of past wars. The criticism of Quetelet's methods is particularly instructive. In connection with Quetelet we may note—without assenting to—one of the Professor's objections to the principle of the "Mean Man." It is in effect the same objection as Cournot raised: that the average of one limb derived from measuring several specimens might not fit the average similarly found as the type of another limb. The model man constructed by putting together these averages of parts might prove to be a monster.

In conclusion we venture to express the hope that this important treatise may be translated into English; in order that the insular student may not have to encounter the difficulties of German and Probabilities at once. We might advise the translator to follow the excellent English custom of prefixing descriptive headings to all the tables. As they stand, a close attention to the context is sometimes required in order to be quite certain of the principle in which figures referred to as "calculated" have been obtained. We refer chiefly to the fourth chapter. There occur also, in the second chapter, some terms vital to the meaning, which may require to be interpreted for the benefit of the English reader; e.g. *Zahlenlotto*, *Klassenlotterie*, *Kontis* relating to the *Sparbank* "Bikuben" in Kopenhagen.

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THEORETICAL PHYSICS IN ITALY.

Trattato di Fisico-Chimica secondo la Teoria Dinamica. Opera Postuma di Enrico dal Pozzo di Mombello. (Milano, 1892.)

THIS is an elementary treatise from the hand of the late Prof. Mombello, of the Free University of Perugia. In the general nature of its contents it might be compared to Prof. Ostwald's *Allgemeine Chemie*. It is, however, much smaller; and is indeed less of a systematic treatise than a condensed statement of the many principles and laws on which physics and chemistry are built. The English terms physics and chemistry have, under the influence of our examination systems, become so stereotyped in meaning that neither term could fitly describe the character of this *Trattato di Fisico-Chimica*. The time-honoured division of subjects would ill fit into its plan. Dynamics, properties of matter, heat, light, sound, electricity, and magnetism are certainly all treated in their more theoretical aspects; but there are also introduced the laws of chemical combination and the atomic theory, which give the book a character possessed by none of our English treatises on physics. A brief sketch of its character may not then be wholly valueless.

The treatise is divided into five parts, under the headings *Dinamica*, *Azione Moleculare*, *Elettrologia*, *Luce e Colore*, *Filosofia Scientifica*.

The first part contains much that we understand as Dynamics; but it contains a good deal more. In chapter I. (*Moto ed Energia*) physics is described as the science of motion. The universal law of nature is the law of causality, after a brief discussion of which we are introduced to four general principles—namely, the law of the conservation of mass, the law of the equality of action and reaction, the rectilinear action of force, and the composition of motions. Then follow two *General Physical Laws*, the conservation of energy and the transformation of energy (*la correlazione ed equivalenza dei moti*). Thereafter are introduced somewhat less general formulæ, which are distinguished as *Definite Physical Laws* and *Definite Chemical Laws*. Of the former, two examples are given—namely, "Pascal's Law" concerning the transmission of pressure in a liquid, and "Dalton's Law" that there is no physical action between the particles of gases, which are not chemically combinable. Then of the definite chemical laws four are particularised, being distinguished by the names of Lavoisier, Proust, Avogadro, and Cannizzaro, the last being a modified statement of Dulong and Petit's law of the specific heats of the elements. The rest of the chapter is devoted to a discussion of inertia, of Newton's laws of motion with special reference to the second interpretation of the third law, of kinetic (*attuale*) and potential energies of action at a distance, and of the conception of stress (*conflitto*) between particles. Chapter II. (*Composizione dei Moti*) is purely kinematical. In chapter III. (*Velocità moleculare*) the physical molecule is led upon the stage. Cohesion, viscosity, rigidity, porosity, volatility, critical points, crystalline form, and gravitation—in a word, the essentially molecular and *dynamic* qualities of bodies—are touched upon; and the whole finishes with an elementary treatment of the kinetic theory of gases, including an account of Crookes's experiments on radiant matter. It is satisfactory to notice that Prof. Mombello, like Prof. Ostwald, has the boldness to speak simply of Boyle's Law untainted by the Marriotte blend. This, of course, is merely historic justice. On the other hand, surely Herapath deserves mention as one of those who aided in the development of the kinetic theory of gases.

This early introduction of the kinetic theory has no doubt its merits; but a more logical course would have been to give in the first place some notion of the real meaning of temperature. This is touched upon in the immediately succeeding chapter, *Teoria termo-dinamica*, which forms Chapter I. of Part II. The treatment here is certainly peculiar. Two theorems (*enunciati*), we are told, are to be taken for the study of heat. The first embraces Carnot's doctrine of the logical necessity for a complete cycle, and his great principle of the reversibility test. Lord Kelvin's definition of temperature is brought in as a kind of corollary and dismissed in a few sentences. We hear no more of Carnot. "The second *enunciato* is concerned with the fact that in the universe an immense indefinite quantity of heat is being generated constantly during the formation of the stars." Then follows a brief sketch of some of the conclusions of spectroscopy, leading up to a broad discussion, in terms of the molecular theory, of the meaning of temperature, and of radiant energy in its four-fold aspect—thermal, luminous, chemical, and phosphorescent. After this thermodynamics, in the usual significance of the term, is presented under the guise of two propositions ascribed to Hirn. These propositions are, to all practical intents and purposes, simply the two laws of thermodynamics. But we search in vain for any reference to Joule; while Rankine and Clausius are merely mentioned as having proposed a demonstration of Hirn's second principle! Now Hirn deserves all credit for his experimental corroboration of the truth that only a fraction of the heat which leaves the boiler is transformed into useful work; but to magnify his labours in the way indicated is surely an inversion of history. Moreover there is no hint as to the relation between Carnot's reversible engine and the second principle; and the absolute zero of temperature is defined only in terms of the gaseous laws. Of entropy and the dissipation of energy we find no trace.

The succeeding chapters of Part II. are devoted to such subjects as the atomic theory (chemical) and the various aspects of capillarity, diffusion, osmose, &c. A brief account is also given of electro-chemistry, although electrical phenomena in general are not discussed till later. The seven chapters of Part III., in which electricity and magnetism are treated, form a highly condensed and instructive compendium of fact and theory, the two not always, perhaps, very clearly distinguished.